Convergence of martingale and moderate deviations for a branching random walk with a random environment in time

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Abstract

We consider a branching random walk on $\mathbb{R}$ with a stationary and ergodic environment $\xi = (\xi_n)$ indexed by time $n \in \mathbb{N}$. Let Z_n be the counting measure of particles of generation n and $\tilde{Z}_n(t) = \int e^{tx} Z_n(dx)$ be its Laplace transform. We show the L^p convergence rate and the uniform convergence of the martingale $\tilde{Z}_n(t)/\mathbb{E}[\tilde{Z}_n(t)|\xi]$, and establish a moderate deviation principle for the measures Z_n.

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1 Introduction

1.1 Model and notation

Branching random walks were largely studied in the literature, see e.g. [5, 6, 7, 8, 9, 13, 28]. In the classical branching random walk, the point processes indexed by the particles u, formulated by the the number of its offsprings and their displacements, have a common distribution for all particles. However, in reality these distributions may differ from generations according to an environment in time, or depend on particles' positions according to an environment in space. For this reason, branching random walks in random environments attract many authors' attention recently. Many results for classical branching random walk have been extended to random environments both in time and space, see e.g. [15, 16, 20, 25, 35, 36]. Here we consider the case in a time random environment, where the distributions of the point processes indexed by particles vary from generation to generation according to a random environment in time. Such a model is called *branching random walk with a random environment in time* (BRWRE). It was first introduced by Biggins & Kyprianou [12]. Recently, some limit theorems such as large deviation principles and central limit theorems were obtained in [19, 25, 26].

Let's describe the model. The random environment in time is modeled as a stationary and ergodic sequence of random variables, $\xi = (\xi_n)$, indexed by the time $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$, taking values in some measurable space $(\Theta, \mathcal{E})$. Without loss of generality we can suppose that ξ is defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}}, \tau)$, with τ the law of ξ. Each realization of ξ_n corresponds to a distribution $\eta_n = \eta(\xi_n)$ on $\mathbb{N} \times \mathbb{R} \times \mathbb{R} \times \cdots$. When the environment $\xi = (\xi_n)$ is given, the process can be described as follows. At time 0, there is an initial particle $\emptyset$ of generation 0 located at $S_\emptyset = 0 \in \mathbb{R}$; at time 1, it is replaced by $N = N(\emptyset)$ particles of generation 1, located at $L_i = L_i(\emptyset)$, $1 \le i \le N$, where the random vector $X(\emptyset) = (N, L_1, L_2, \cdots) \in \mathbb{N}\times\mathbb{R}\times\mathbb{R}\times\cdots$ is of distribution $\eta_0 = \eta(\xi_0)$. In general, each particle $u = u_1 \cdots u_n$ of generation n located at S_u is replaced at time n+1 by $N(u)$ new particles ui of generation $n+1$, located at

$$S_{ui} = S_u + L_i(u) \qquad (1 \le i \le N(u)),$$

where the random vector $X(u) = (N(u), L_1(u), L_2(u), \cdots)$ is of distribution $\eta_n = \eta(\xi_n)$. Note that the values $L_i(u)$ for $i > N(u)$ do not play any role for our model; we introduce them only for convenience. We can for example take $L_i(u) = 0$ for $i > N(u)$. All particles behave independently conditioned on the environment ξ.

For each realization $\xi \in \Theta^{\mathbb{N}}$ of the environment sequence, let $(\Gamma, \mathcal{G}, \mathbb{P}_\xi)$ be the probability space under which the process is defined. The probability $\mathbb{P}_\xi$ is usually called *quenched law*. The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, \mathcal{E}^{\mathbb{N}} \otimes \mathcal{G}, \mathbb{P})$, where $\mathbb{P} = \mathbb{E}(\delta_\xi \otimes \mathbb{P}_\xi)$ with δ_ξ the Dirac measure at ξ and $\mathbb{E}$ the expectation with respect to the law of ξ, so that for all measurable and positive function g defined on $\Theta^{\mathbb{N}} \times \Gamma$, we have

$$\int_{\Theta^{\mathbb{N}} \times \Gamma} g(x, y) d\mathbb{P}(x, y) = \mathbb{E} \int_\Gamma g(\xi, y) d\mathbb{P}_\xi(y).$$

The total probability $\mathbb{P}$ is usually called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be considered to be the conditional probability of $\mathbb{P}$ given ξ. The expectation with respect to $\mathbb{P}$ will still be denoted by $\mathbb{E}$; there will be no confusion for reason of consistence. The expectation with respect to $\mathbb{P}_\xi$ will be denoted by $\mathbb{E}_\xi$.

Let

$$\mathbb{U} = \{\emptyset\} \bigcup_{n \geq 1} \mathbb{N}^n$$

be the set of all finite sequence $u = u_1 \cdots u_n$. By definition, under $\mathbb{P}_\xi$, the random vectors $\{X(u)\}$, indexed by $u \in \mathbb{U}$, are independent of each other, and each $X(u)$ has distribution $\eta_n = \eta(\xi_n)$ if $|u| = n$, where $|u|$ denotes the length of u. Let $\mathbb{T}$ be the Galton-Watson tree with defining element $\{N(u)\}$. We have: (a) $\emptyset \in \mathbb{T}$; (b) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \leq i \leq N(u)$; (c) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$. Let $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of particles of generation $n \in \mathbb{N}$ and

$$Z_n(\cdot) = \sum_{u \in \mathbb{T}_n} \delta_{S_u}(\cdot) \tag{1.1}$$

be the counting measure of particles of generation n. For a measurable subset A of $\mathbb{R}$, $Z_n(A)$ denotes the number of particles of generation n located in A. For any finite sequence u, let

$$X^{(u)}(\cdot) = \sum_{i=1}^{N(u)} \delta_{L_i(u)}(\cdot) \tag{1.2}$$

be the counting measure corresponding to the random vector $X(u)$, whose increasing points are $L_i(u)$, $1 \leq i \leq N(u)$. Denote $u|n$ by the restriction to the first n terms of u, with the convention that $u_0|0 = \emptyset$. Set

$$X_n = X^{(u_0|n)}, \tag{1.3}$$

where $u_0 = (1, 1, \cdots)$. The counting measure X_n describes the evolution of the system at time n.

For $n \in \mathbb{N}$ and $t \in \mathbb{R}$, denote

$$\tilde{Z}_n(t) = \int e^{tx} Z_n(dx) = \sum_{u \in \mathbb{T}_n} e^{tS_u} \tag{1.4}$$

the Laplace transform of Z_n. It is also called *partition function* by physicians. In particular, for $t = 0$, $\tilde{Z}_n(0) = Z_n(\mathbb{R})$. Let

$$m_n(t) = \mathbb{E}_\xi \int e^{tx} X_n(dx) = \mathbb{E}_\xi \sum_{i=1}^{N(u)} e^{tL_i(u)} \quad (|u| = n), \tag{1.5}$$

be the Laplace transform of the counting measure describing the evolution of the system at time n. Put

$$P_0(t) = 1 \qquad \text{and} \qquad P_n(t) = \prod_{i=0}^{n-1} m_i(t) \quad \text{for } n \geq 1. \tag{1.6}$$

Then $P_n(t) = \mathbb{E}_\xi \tilde{Z}_n(t)$. Moreover, set

$$\tilde{X}_u(t) = \frac{e^{tS_u}}{P_n(t)} \qquad (|u| = n), \tag{1.7}$$

and

$$W_n(t) = \frac{\tilde{Z}_n(t)}{P_n(t)} = \sum_{u \in \mathbb{T}_n} \tilde{X}_u(t). \tag{1.8}$$

Let $\mathcal{F}_0 = \sigma(\xi_0, \xi_1, \xi_2, \cdots)$ and $\mathcal{F}_n = \sigma(\xi_0, \xi_1, \xi_2, \cdots, X(u),\ |u| < n,\ i = 1, 2, \cdots)$. It is well known that for each t fixed, $W_n(t)$ forms a nonnegative martingale with respect to the filtration $\mathcal{F}_n$ under both laws $\mathbb{P}_\xi$ and $\mathbb{P}$, and

$$\lim_{n\to\infty} W_n(t) = W(t) \qquad a.s. \tag{1.9}$$

with $\mathbb{E}_\xi W(t) \le 1$. In the deterministic environment case, this martingale has been studied by Kahane & Peyrière [27], Biggins [5], Durrett & Liggett [18], Guivarc'h [21], Lyons [34] and Liu [30, 31, 32, 33], etc. in different contexts.

Assume throughout that

$$\mathbb{E} \log m_0(0) \in (0, \infty) \qquad \text{and} \qquad \mathbb{E}\left[\frac{N}{m_0(0)} \log^+ N\right] < \infty. \tag{1.10}$$

The first condition means that the corresponding branching process in a random environment (BPRE), $\{Z_n(\mathbb{R})\}$, is *supercritical*, so that the survival of the population $\{Z_n(\mathbb{R}) \to \infty\}$ has positive probability; and the two conditions ensure that the limit of the normalized population, $W(0)$, is non-degenerate (cf. [2, 3]). We also assume that

$$\mathbb{E}|\log m_0(t)| < \infty \qquad \text{and} \qquad \mathbb{E}\left|\frac{m_0'(t)}{m_0(t)}\right| < \infty \tag{1.11}$$

for all $t \in \mathbb{R}$. The last two moment conditions imply that

$$\Lambda(t) := \mathbb{E} \log m_0(t) \qquad \text{and} \qquad \Lambda'(t) = \mathbb{E}\frac{m_0'(t)}{m_0(t)} \tag{1.12}$$

are well defined as real numbers, so that $\Lambda(t)$ is differentiable everywhere on $\mathbb{R}$ with $\Lambda'(t)$ as its derivative. Let

$$\begin{aligned} t_- &= \inf\{t \in \mathbb{R} : t\Lambda'(t) - \Lambda(t) \le 0\}, \\ t_+ &= \sup\{t \in \mathbb{R} : t\Lambda'(t) - \Lambda(t) \le 0\}, \end{aligned} \tag{1.13}$$

Then $-\infty \le t_- < 0 < t_+ \le \infty$, t_- and t_+ are two solutions of $t\Lambda'(t) - \Lambda(t) = 0$ if they are finite. Denote

$$I = (t_-, t_+).$$

If $t \in I$ and $\mathbb{E}W_1(t) \log^+ W_1(t) < \infty$, then $\mathbb{E}_\xi W(t) = 1$ a.s. (cf. [12, 29]).

1.2 L^p convergence rate

We first study the L^p ($p > 1$) convergence of $W_n(t)$ to its limit $W(t)$ and its exponential rate for $t \in \mathbb{R}$ fixed. When $t = 0$, $W_n(0)$ reduces to the normalized population of the corresponding BPRE, whose convergence rate is carefully discussed in Huang & Liu [24]. Without loss of generality, here we only consider the case where $t = 1$ and assume that

$$m_0(1) = 1. \tag{1.14}$$

Write $W_n = W_n(1)$ for short. For general case, if $m_0(t) \in (0, \infty)$ a.s., we can construct a new BRERE with relative displacements $\bar{L}_i(u) = tL_i(u) - \log m_n(t)$ ($|u| = n$). Then this new BRERE satisfies $\bar{m}_0(1) = 1$ and $\bar{W}_n = W_n(t)$. Furthermore, we also assume that

$$\mathbb{P}(W_1 = 1) < 1, \tag{1.15}$$

which avoids the trivial case where $W_n = 1$ a.s..

The following theorem shows the L^p convergence (with $\rho = 1$) of W_n under quenched law $\mathbb{P}_\xi$ and its exponential rate (with $\rho > 1$).

Theorem 1.1 (Quenched L^p convergence rate)**.** *Assume (1.14). Let $p > 1$ and $\rho \ge 1$.*

(a) If $1 < p < 2$,

$$\mathbb{E} \log \mathbb{E}_\xi W_1^r < \infty \qquad \textit{and} \qquad \rho < \exp(-\frac{1}{r}\mathbb{E} \log m_0(r))$$

for some $r \in [p, 2]$, then

$$W_n - W = o(\rho^{-n}) \quad \textit{a.s. and in } \mathbb{P}_\xi\text{-}L^p \textit{ for almost all } \xi.$$

(b) If $p=2$ or $2<p\le t_+$, and $\mathbb{E}\log\mathbb{E}_\xi W_1^p<\infty$, then for for almost all ξ,

$$\limsup_{n\to\infty}\rho^n(\mathbb{E}_\xi|W_n-W|^p)^{1/p}\begin{cases} =0 & \text{if } \rho<\rho_c,\\ >0 & \text{if } \rho>\rho_c \text{ and } \mathbb{E}\log^-\mathbb{E}_\xi|W_1-1|^2<\infty,\end{cases}$$

where $\rho_c=\exp(-\frac{1}{2}\mathbb{E}\log m_0(2))$.

If $p=2$ or $2<p\le t_+$, Theorem 1.1(b) shows that under certain moment conditions, the value ρ_c is the critical value for the L^p convergence of $\rho^n(W_n-W)$ to 0 under quenched law $\mathbb{P}_\xi$. In order to show the L^p convergence of W_n under annealed law $\mathbb{P}$ and its rate, we need to assume that the environment random variables (ξ_n) are i.i.d..

Theorem 1.2 (Annealed L^p convergence)**.** *Assume (1.14), (1.15) and that (ξ_n) are i.i.d.. Let $p>1$. Then $W_n\to W$ in $\mathbb{P}$-L^p if and only if*

$$\mathbb{E}W_1^p<\infty \qquad \text{and} \qquad \mathbb{E}m_0(p)<1.$$

Theorem 1.2 coincide with a result of Liu [32] on branching random walk in a deterministic environment, and is an extension of a result of Guivarc'h & Liu [22] on branching process in a random environment. The same result is obtained in [25] with a different approach.

For the exponential rate of the annealed L^p convergence of W_n, we have the follow result.

Theorem 1.3 (Annealed L^p convergence rate)**.** *Assume (1.14), (1.15) and that (ξ_n) are i.i.d.. Let $p>1$ and $\rho>1$.*

(a) If $1<p<2$,

$$\mathbb{E}(\mathbb{E}_\xi W_1^r)^{p/r}<\infty \qquad \text{and} \qquad \rho<[\mathbb{E}m_0(r)^{p/r}]^{-1/p}$$

for some $r\in[p,2]$, then

$$W_n-W=o(\rho^{-n}) \qquad \text{in } \mathbb{P}\text{-}L^p.$$

(b) If $p\ge 2$ and $\mathbb{E}W_1^p<\infty$, then

$$\limsup_{n\to\infty}\rho^n(\mathbb{E}|W_n-W|^p)^{1/p}\begin{cases} =0 & \text{if } \rho<\rho_0,\\ >0 & \text{if } \rho>\rho_0,\end{cases}$$

where $\rho_0=\min\{[\mathbb{E}m_0(p)]^{-1/p},[\mathbb{E}m_0(2)^{p/2}]^{-1/p}\}$.

Under the conditions of Theorem 1.3, we can also obtain $W_n-W=o(\rho^{-n})$ a.s. in $\mathbb{P}_\xi$-L^p for almost all ξ. However, by Jensen's inequality, one can see that $\frac{r}{p}\mathbb{E}\log\mathbb{E}_\xi W_1^r\le\log\mathbb{E}(\mathbb{E}_\xi W_1^r)^{p/r}$ and $[\mathbb{E}m_0(r)^{p/r}]^{-1/p}\le\exp(-\frac{1}{r}\mathbb{E}\log m_0(r))$. So the moment conditions of Theorem 1.1 are weaker than those of Theorem 1.3. If $p\ge 2$, Theorem 1.3 (b) shows that under the moment condition $\mathbb{E}W_1^p<\infty$, the value ρ_0 defined above is the critical value for the L^p convergence of $\rho^n(W_n-W)$ to 0 under annealed law $\mathbb{P}$. Obviously, we have $\rho_0\le\rho_c$. But here it is a pity that we do not find the critical value for $p\in(1,2)$, in contrast to [24] for branching process in a random environment.

1.3 Uniform convergence

We next consider the uniform convergence of $W_n(t)$ to its limit $W(t)$. In the deterministic environment case, such result was shown by Biggins [10, 11] and recently is generalized by Attia [4].

Recall that $I=(t_-,t_+)$, where t_+ and t_- are defined by (1.13). If $t\in I$ and $\mathbb{E}W_1(t)\log^+W_1(t)<\infty$, then $W(t)$ is non-degenerate. Similarly to [10, 11, 4], we consider the uniform convergence of $W_n(t)$ on subsets of I. Denote

$$\begin{aligned}\underline{m}_0&=\inf_{t\in I}m_0(t),\\ \Omega_1&=int\{t\in\mathbb{R}:\mathbb{E}\log\mathbb{E}_\xi W_1(t)^\gamma \text{ for some } \gamma>1\},\\ \Omega_2&=int\{t\in\mathbb{R}:\mathbb{E}\tilde{Z}_1(t)\log^+\tilde{Z}_1(t)<\infty\}.\end{aligned} \tag{1.16}$$

Here and after we use the following usual notations:

$$\log^+x=\max(\log x,0),\quad \log^-x=\max(-\log x,0).$$

Theorem 1.4 (Quenched uniform convergence). *Assume that* $\mathbb{E}\log^- \underline{m}_0 < \infty$.

(a) Let K *be a compact subset of* $I \bigcap \Omega_1$*. Then there exists a constant* $p_K \in (1,2]$ *such that* $W_n(t)$ *converges uniformly to* $W(t)$ *on* K*, almost surely and in* $\mathbb{P}_\xi$-L^p *for almost all* ξ *if* $p \in [1, p_K]$.

(b) If $\underline{m} := essinf \underline{m}_0 > 0$*, then* $W_n(t)$ *also converges uniformly to* $W(t)$ *on any compact subset* K *of* $I \bigcap \Omega_2$*, almost surely and in* $\mathbb{P}_\xi$-L^1 *for almost all* ξ.

If the environment random variables (ξ_n) are i.i.d., Theorem 1.4 have the following comparison under annealed law.

Theorem 1.5 (Annealed uniform convergence). *Assume that* $\underline{m} = essinf \underline{m}_0 > 0$*. Denote*

$$I' = int\{t \in \mathbb{R} : \mathbb{E}\left[\frac{m_0(\gamma t)}{m_0(t)^\gamma}\right] < 1 \;\; for\ some\ \gamma > 1\}.$$

Then $W_n(t)$ *converges uniformly to* $W(t)$ *on any compact subset* K *of* $I \bigcap I' \bigcap \Omega_2$ *in* $\mathbb{P}$-L^1.

Moreover, set

$$\Omega_1' = int\{t \in \mathbb{R} : \mathbb{E}\tilde{Z}_1(t)^\gamma < \infty \;\; for\ some\ \gamma > 1\}.$$

If K *is a compact subset of* $I \bigcap \Omega_1'$*, then there exists a constant* $p_K \in (1,2]$ *such that* $W_n(t)$ *converges uniformly to* $W(t)$ *on* K *in* $\mathbb{P}$-L^{p_K}.

It is clear that $\Omega_1' \subset \Omega_2$, but there is no evident relation between I and I'. By calculating the derivative of $\mathbb{E}\left[\frac{m_0(\gamma t)}{m_0(t)^\gamma}\right]$ with respect to γ and letting $\gamma = 1$, we can see that $I \bigcap \Omega_1' \subset I' \bigcap \Omega_1'$.

1.4 Moderate deviation

Finally we state a moderate deviation principle about the counting measures Z_n. Recently, Huang & Liu showed the large deviation principle ([26] , Theorem 3.2) and central limit theorem ([26] , Theorem 7.1) about Z_n, which reflect the asymptotic properties of normalized measure $\frac{Z_n(n\cdot)}{Z_n(\mathbb{R})}$ and $\frac{Z_n(b_n\cdot)}{Z_n(\mathbb{R})}$ (with some b_n satisfies that $b_n/\sqrt{n}$ goes to a positive limit) . We want to establish the corresponding moderate deviation principle.

Let (a_n) be a sequence of positive numbers satisfying

$$\frac{a_n}{n} \to 0 \qquad \text{and} \qquad \frac{a_n}{\sqrt{n}} \to \infty. \tag{1.17}$$

We are interested in the asymptotic properties of normalized measure $\frac{Z_n(a_n\cdot)}{Z_n(\mathbb{R})}$.

Theorem 1.6 (Moderate deviation principle). *Write* $\pi_0 = m_0(0)$*. Assume that* $\|\frac{1}{\pi_0}\mathbb{E}_\xi \sum_{i=1}^N e^{\delta|L_i|}\|_\infty := esssup \frac{1}{\pi_0}\mathbb{E}_\xi \sum_{i=1}^N e^{\delta|L_i|} < \infty$ *for some* $\delta > 0$ *and* $\mathbb{E}_\xi \sum_{i=1}^N L_i = 0$ *a.s.. If* $0 \in \Omega_1$*, then the sequence of finite measures* $A \mapsto Z_n(a_n A)$ *satisfies a principle of moderate deviation with rate function* $\frac{x^2}{2\sigma^2}$*: for each measurable subset* A *of* $\mathbb{R}$*,*

$$\begin{aligned} -\frac{1}{2\sigma^2}\inf_{x\in A^o} x^2 \quad &\leq \quad \liminf_{n\to\infty} \frac{n}{a_n^2} \log \frac{Z_n(a_n A)}{Z_n(\mathbb{R})} \\ &\leq \quad \limsup_{n\to\infty} \frac{n}{a_n^2} \log \frac{Z_n(a_n A)}{Z_n(\mathbb{R})} \leq -\frac{1}{2\sigma^2}\inf_{x\in \bar{A}} x^2 \end{aligned}$$

a.s. on $\{Z_n(\mathbb{R}) \to \infty\}$*, where* $\sigma^2 = \mathbb{E}\left[\frac{1}{\pi_0}\sum_{i=1}^N L_i^2\right]$*,* A^o *denotes the interior of* A*, and* $\bar{A}$ *its closure.*

The rest part of the paper is arranged as follows. We first study the L^p convergence and its exponential rate of the martingale W_n in Section 2 under quenched law and in Section 3 under annealed law. Then we prove the uniform convergence of the martingale $W_n(t)$ in Section 4. Finally, in Section 5, we consider moderate deviations related to the counting measures Z_n.

2 Quenched L^p convergence; proof of Theorem 1.1

In this section, we shall study the L^p convergence of W_n under quenched law $\mathbb{P}_\xi$ and its exponential rate. To prove the results about the quenched convergence, we need the following lemma.

Lemma 2.1. *([24], Lemma 3.1) Let $(\alpha_n, \beta_n)_{n\geq 0}$ be a stationary and ergodic sequence of non-negative random variables. If $\mathbb{E}\log\alpha_0 < 0$ and $\mathbb{E}\log^+\beta_0 < \infty$, then*

$$\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1}\beta_n < \infty \quad a.s.. \tag{2.1}$$

Conversely, if $\mathbb{E}|\log\beta_0| < \infty$, then (2.1) implies that $\mathbb{E}\log\alpha_0 \leq 0$.

Recall that $W_n = W_n(1)$. To estimate the exponential rate of W_n, we consider the series introduced by Alsmeyer et al. [1]:

$$A = A(\rho) = \sum_{n=0}^{\infty} \rho^n (W - W_n) \quad (\rho > 1), \tag{2.2}$$

$$\hat{A}_n = \hat{A}_n(\rho) = \sum_{k=0}^{n} \rho^k (W_{k+1} - W_k) \quad (\rho \geq 1), \tag{2.3}$$

$$\hat{A} = \hat{A}(\rho) = \sum_{n=0}^{\infty} \rho^n (W_{n+1} - W_n) = \lim_{n\to\infty} \hat{A}_n \quad (\rho \geq 1). \tag{2.4}$$

According to ([1], Lemma 3.1), with the same $\rho > 1$, A and $\hat{A}$ have the same convergence in the sense a.s. and in L^p under $\mathbb{P}_\xi$ or $\mathbb{P}$. Since W_n is a martingale under both laws $\mathbb{P}_\xi$ and $\mathbb{P}$, the same is true for $\hat{A}_n$ (but with respect to the filtration $\mathcal{F}_{n+1}$). In particular, if $\rho = 1$, one can see that $\hat{A}_n = W_{n+1} - 1$. Therefore we can study the convergence of $\hat{A}$ by Doob's convergence theorems for martingales, which means that we should show a uniform upper bound for the p-th moment of $\hat{A}_n$ under $\mathbb{P}_\xi$ for quenched case and under $\mathbb{P}$ for annealed case. To this end, we will use Bukholder's inequality as the basic tool. We mention that our approaches are very similar to Huang & Liu [24] and Alsmeyer et al. [1], but the method of measure change for the annealed case would be heuristic.

Lemma 2.2 (Burkholder's inequality, see e.g. [14])**.** *Let $\{S_n\}$ be a L^1 martingale with $S_0 = 0$. Let $Q_n = (\sum_{k=1}^{n}(S_k - S_{k-1})^2)^{1/2}$ and $Q = (\sum_{n=1}^{\infty}(S_n - S_{n-1})^2)^{1/2}$. Then $\forall p > 1$,*

$$c_p \parallel Q_n \parallel_p \leq \parallel S_n \parallel_p \leq C_p \parallel Q_n \parallel_p,$$

$$c_p \parallel Q \parallel_p \leq \sup_n \parallel S_n \parallel_p \leq C_p \parallel Q \parallel_p,$$

where $c_p = (p-1)/18p^{3/2}$, $C_p = 18p^{3/2}/(p-1)^{1/2}$.

Applying Burkholder's inequality, we can obtain the moment results of $\hat{A}_n$ for $1 < p \leq 2$.

Proposition 2.3 (Quenched moments of $\hat{A}_n$: case $1 < p \leq 2$)**.** *Assume (1.14). Let $1 < p \leq 2$ and $\rho \geq 1$. If $\mathbb{E}\log\mathbb{E}_\xi W_1^r < \infty$ and $\rho < \exp(-\frac{1}{r}\mathbb{E}\log m_0(r))$ for some $r \in [p, 2]$, then $\sup_n \mathbb{E}_\xi|\hat{A}_n|^p < \infty$ a.s.*

Proof. Notice that

$$W_{n+1} - W_n = \sum_{u\in\mathbb{T}_n} \tilde{X}_u (W_{1,u} - 1),$$

where we write $\tilde{X}_u = \tilde{X}_u(1)$ for short, and under quenched law $\mathbb{P}_\xi$, $\{W_{k,u}(t)\}_{|u|=n}$ are i.i.d. and independent of $\mathcal{F}_n$ with common distribution determined by $\mathbb{P}_\xi(W_{k,u}(t) \in \cdot) = \mathbb{P}_{T^n\xi}(W_k(t) \in \cdot)$. The notation T represents the shift operator: $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$ if $\xi = (\xi_0, \xi_1, \cdots)$. Applying Burkholder's inequality to $W_{n+1} - W_n$, and noticing the concavity of $x^{p/2}$, $x^{r/2}$ and $x^{p/r}$, we have

$$\begin{aligned} \mathbb{E}_\xi|W_{n+1} - W_n|^p &\leq C\mathbb{E}_\xi \left(\sum_{u\in\mathbb{T}_n} \tilde{X}_u^2 (W_1(u) - 1)^2\right)^{p/2} \\ &\leq C\left(\mathbb{E}_\xi \sum_{u\in\mathbb{T}_n} \tilde{X}_u^r |W_1(u) - 1|^r\right)^{p/r} \\ &= CP_n(r)^{p/r} \left(\mathbb{E}_{T^n\xi}|W_1 - 1|^r\right)^{p/r}, \end{aligned} \tag{2.5}$$

where C is positive constant, and in general, it does not stand for the same constant throughout. Noticing (2.5), and applying again Burkholder's inequality to $\hat{A}_n$ gives

$$\begin{aligned}\sup_n \mathbb{E}_\xi|\hat{A}_n|^p &\leq C\mathbb{E}_\xi\left(\sum_{n=0}^{\infty}\rho^{2n}(W_{n+1}-W_n)^2\right)^{p/2}\\ &\leq C\sum_{n=0}^{\infty}\rho^{pn}\mathbb{E}_\xi|W_{n+1}-W_n|^p\\ &\leq C\sum_{n=0}^{\infty}\rho^{pn}P_n(r)^{p/r}(\mathbb{E}_{T^n\xi}|W_1-1|^r)^{p/r}.\end{aligned}$$

Since $\mathbb{E}\log\mathbb{E}_\xi W_1^r<\infty$ and $\log\rho+\mathbb{E}\log m_0(r)/r<0$, by Lemma 2.1, the series $\sum_n \rho^{pn}P_n(r)^{p/r}(\mathbb{E}_{T^n\xi}|W_1-1|^r)^{p/r}$ converges a.s., which leads to $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s.. □

For $p>2$, we also have results for the quenched moments of $\hat{A}_n$.

Proposition 2.4 (Quenched moments of $\hat{A}_n$: case $p>2$)**.** *Assume (1.14). Let $p\geq 2$ and $\rho\geq 1$.*

(a) If $2<p\leq t_+$, $\mathbb{E}\log\mathbb{E}_\xi W_1^p<\infty$ and $\rho<\exp(-\frac{1}{2}\mathbb{E}\log m_0(2))$, then $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s..

(b) If $\mathbb{E}|\log\mathbb{E}_\xi|W_1-1|^2|<\infty$, then $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s. implies that $\rho\leq\exp(-\frac{1}{2}\mathbb{E}\log m_0(2))$.

The proof of Proposition 2.4 is based on the following result about the moment of W_n.

Lemma 2.5. *Fix $t>0$. If $1<p\leq\frac{t_+}{t}$ and $\mathbb{E}\log\mathbb{E}_\xi W_1(t)^p<\infty$, then*

$$\sup_n\mathbb{E}_\xi W_n(t)^p<\infty \qquad a.s.. \tag{2.6}$$

Proof. Assume that $p\in(2^b,2^{b+1}]$ for some integer $b\geq 0$. We will prove (2.6) by induction on b. Firstly, for $b=0$, we have $1<p\leq 2$. Recall that $\Lambda(t)=\mathbb{E}\log m_0(t)$. For $t>0$ fixed, set $h_t(x)=\frac{1}{x}\Lambda(tx)$, whose derivative is $h_t'(t)=\frac{1}{x^2}(tx\Lambda'(tx)-\Lambda(tx))$. Since $h_t'(x)<0$ on $(\frac{t_-}{t},\frac{t_+}{t})$, the function $h_t(x)$ is strictly decreasing on $[\frac{t_-}{t},\frac{t_+}{t}]$. Thus

$$\frac{1}{p}\mathbb{E}\log\frac{m_0(pt)}{m_0(t)^p}=h_t(p)-h_t(1)<0.$$

Noticing $\mathbb{E}\log\mathbb{E}_\xi W_1(t)^p<\infty$, we obtain (2.6) by applying Proposition 2.3 with $\rho=1$.

Now suppose the conclusion holds for $p\in(2^b,2^{b+1}]$. For $p\in(2^{b+1},2^{b+2}]$, we have $p/2\in(2^b,2^{b+1}]$. Observe that

$$\mathbb{E}_\xi W_1(2t)^{p/2}=\mathbb{E}_\xi\left(\sum_{u\in\mathbb{T}_1}\frac{e^{2tS_u}}{m_0(2t)}\right)^{p/2}\leq\mathbb{E}_\xi\left[\frac{\left(\sum_{u\in\mathbb{T}_1}e^{tS_u}\right)^p}{m_0(2t)^{p/2}}\right]=\mathbb{E}_\xi W_1(t)^p\frac{m_0(t)^p}{m_0(2t)^{p/2}}.$$

It follows that $\mathbb{E}\log\mathbb{E}_\xi W_1(t)^p<\infty$ implies $\mathbb{E}\log\mathbb{E}_\xi W_1(2t)^{p/2}<\infty$. As $1<\frac{p}{2}\leq\frac{t_+}{2t}$, by induction, we have

$$\sup_n\mathbb{E}_\xi W_n(2t)^{p/2}<\infty \qquad a.s.. \tag{2.7}$$

Notice that by Burkholder's inequality and Minkowski's inequality,

$$\sup_n\mathbb{E}_\xi|W_n(t)-1|^p\leq C\left(\sum_{n=0}^{\infty}(\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|^p)^{2/p}\right)^{p/2}.$$

Using Burkholder's inequality and Jensen's inequality, we have

$$\begin{aligned}
\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|^p &\leq C\mathbb{E}_\xi\left(\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(t)(W_{1,u}(t)-1)^2\right)^{p/2}\\
&\leq C\mathbb{E}_\xi\left(\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(t)\right)^{p/2-1}\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(t)|W_{1,u}(t)-1|^p\\
&= C\mathbb{E}_\xi\left(\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(t)\right)^{p/2}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\\
&= C\frac{P_n(2t)^{p/2}}{P_n(t)^p}\mathbb{E}_\xi W_n(2t)^{p/2}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p. \qquad (2.8)
\end{aligned}$$

Noticing (2.7), to get (2.6), it suffices to show the convergence of the series

$$\sum_n\frac{P_n(2t)}{P_n(t)^2}\left(\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\right)^{2/p}. \qquad (2.9)$$

Since $1<2<p\leq\frac{t_+}{t}$, we have

$$\mathbb{E}\log m_0(t)=h_t(1)>h_t(2)=\frac{1}{2}\mathbb{E}\log m_0(2t).$$

By Lemma 2.1, the series (2.9) converges a.s.. The proof is complete. □

Proof of Proposition 2.4. We first consider the assertion (a). The conclusion for $\rho=1$ is contained in Lemma 2.5. For $\rho>1$, similarly to the proof of Lemma 2.5, applying Burkholder's inequality to $\hat{A}_n$ and noticing (2.8), the conclusion follows by the convergence of the series

$$\sum_n\rho^{2n}P_n(2)(\mathbb{E}_\xi W_n(2)^{p/2})^{2/p}\left(\mathbb{E}_{T^n\xi}|W_1-1|^p\right)^{2/p}. \qquad (2.10)$$

Since $\mathbb{E}\log\mathbb{E}_\xi W_1^p<\infty$ (so that $\mathbb{E}\log\mathbb{E}_\xi W_1(2)^{p/2}<\infty$) and $1<\frac{p}{2}\leq\frac{t_+}{2}$, Lemma 2.5 gives $\sup_n\mathbb{E}_\xi W_n(2)^{p/2}<\infty$ a.s.. By Lemma 2.1, the series (2.10) converges a.s. if $\rho<\exp(-\frac{1}{2}\mathbb{E}\log m_0(2))$.

We next consider the assertion (b). By Burkholder's inequality,

$$\begin{aligned}
\sup_n\mathbb{E}_\xi|\hat{A}_n|^p &\geq C\sum_{n=0}^\infty\rho^{pn}\mathbb{E}_\xi|W_{n+1}-W_n|^p\\
&\geq C\sum_{n=0}^\infty\rho^{pn}\mathbb{E}_\xi\left(\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(W_{1,u}-1)^2\right)^{p/2}\\
&\geq C\sum_{n=0}^\infty\rho^{pn}\left(\mathbb{E}_\xi\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(W_{1,u}-1)^2\right)^{p/2}\\
&= C\sum_{n=0}^\infty\rho^{pn}P_n(2)^{p/2}(\mathbb{E}_{T^n\xi}|W_1-1|^2)^{p/2}.
\end{aligned}$$

Since $\mathbb{E}|\log\mathbb{E}_\xi|W_1-1|^2|<\infty$, we deduce $\rho\leq\exp(-\frac{1}{2}\mathbb{E}\log m_0(2))$ by Lemma 2.1. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. For the assertion (a), by Proposition 2.3, we have $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s., which implies that $\rho^n(W_n-W)\to0$ in $\mathbb{P}_\xi$-L^p for almost all ξ. For the assertion (b), if $\rho<\rho_c$, by Proposition 2.4(a), we have $\sup_n\mathbb{E}_\xi|\hat{A}_n|^p<\infty$ a.s., so that $\rho^n(W-W_n)\to0$ in $\mathbb{P}_\xi$-L^p for almost all ξ. If $\rho>\rho_c$ and $\mathbb{E}\log^-\mathbb{E}_\xi|W_1-1|^2<\infty$, we assume that $\rho^n(\mathbb{E}|W-W_n|^p)^{1/p}\to0$ a.s.. Denote $D=\{\xi:\rho^n(\mathbb{E}|W-W_n|^p)^{1/p}\to0\}$. Following similar argument in ([24], proof of Theorem 1.2), we can see that $\mathbb{P}(D)=0$ or 1.

If $\mathbb{P}(D)=1$, the sequence $\rho^n(\mathbb{E}_\xi|W-W_n|^p)^{1/p}$ is bounded for almost all ξ. Denote this bound by $M(\xi)$. For any $1<\rho_1<\rho$, the series $\sum_n \rho_1^n(W-W_n)$ converges in $\mathbb{P}_\xi$-L^p since

$$\left(\mathbb{E}_\xi\left|\sum_n \rho_1^n(W-W_n)\right|^p\right)^{1/p} \le \sum_n \rho_1^n(\mathbb{E}_\xi|W-W_n|^p)^{1/p} \le M(\xi)\sum_n\left(\frac{\rho_1}{\rho}\right)^n<\infty \quad a.s..$$

Thus $A(\rho_1)$ and $\hat{A}(\rho_1)$ converge in $\mathbb{P}_\xi$-L^p, so that $\sup_n \mathbb{E}_\xi|\hat{A}_n(\rho_1)|^p<\infty$ a.s.. By Proposition 2.4(b), we have $\rho_1\le\rho_c$. Letting $\rho_1\uparrow\rho$ yields $\rho\le\rho_c$. This contradicts the fact that $\rho>\rho_c$. Thus $\mathbb{P}(D)=0$, i.e., $\rho^n(\mathbb{E}_\xi|W-W_n|^p)^{1/p}\nrightarrow 0$ for almost all ξ. □

3 Annealed L^p convergence

3.1 Change of measure

Inspired by the idea of the classic measure change (see for example Lyons [34], Biggins & Kyprianou [12], Hu & Shi [23]), we introduce a new probability measure as follows.

When the environment ξ is given, for $t\in\mathbb{R}$ fixed, define a new probability $\mathbb{Q}_\xi=\mathbb{Q}_\xi^{(t)}$ such that for any $n\ge1$,

$$\mathbb{Q}_\xi|_{\mathcal{F}_n}=W_n(t)\mathbb{P}_\xi|_{\mathcal{F}_n}. \tag{3.1}$$

The existence of $\mathbb{Q}_\xi$ is ensured by Kolmogorov's extension theorem. In fact, under $\mathbb{Q}_\xi$, the tree $\mathbb{T}$ is a so-called size-biased weighted tree (see for example Kuhlbusch (2004, [29]) for the construction of a size-biased tree).

Fix $n\ge1$. Let $\omega_n^n=\omega_n^n(t)$ be a random variable taking values in $\mathbb{T}_n$ such that for any $u\in\mathbb{T}_n$,

$$\mathbb{Q}_\xi(\omega_n^n=u|\mathcal{F}_\infty)=\frac{\tilde{X}_u(t)}{W_n(t)}, \tag{3.2}$$

where $\mathcal{F}_\infty=\sigma(\mathcal{F}_n,n\ge0)$. Denote $\omega_k^n=\omega_n^n|k$ for $k=0,1,\cdots,n$. So $(\omega_0^n,\omega_1^n,\cdots,\omega_n^n)$ is the vertices visited by the shortest path in $\mathbb{T}$ connecting the root $\omega_0^n=\emptyset$ with ω_n^n.

Lemma 3.1. *Fix $t\in\mathbb{R}$ and $n\ge1$. For all nonnegative Borel functions h and g (defined on $\mathbb{R}$ or $\mathbb{R}^2$), we have for each $k=1,2,\cdots,n$,*

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_\xi} h\left(\tilde{X}_{\omega_k^n}(t),\sum_{\substack{v\in\mathbb{T}_k\\ v\ne\omega_k^n}}\tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,\omega_k^n}(t)\right)\\
=\ &\mathbb{E}_\xi\sum_{v\in\mathbb{T}_k}\tilde{X}_u(t)h\left(\tilde{X}_u(t),\sum_{\substack{v\in\mathbb{T}_k\\ v\ne u}}\tilde{X}_v(t)W_{n-k,v}(t)\right)\mathbb{E}_{T^k\xi}W_{n-k}(t)g\left(W_{n-k}(t)\right).
\end{aligned} \tag{3.3}$$

Proof. By the definition of ω_k^n and $\mathbb{Q}_\xi$, we can calculate that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_\xi} h\left(\tilde{X}_{\omega_k^n}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,\omega_k^n}(t)\right)\\
= \;& \mathbb{E}_{\mathbb{Q}_\xi} \sum_{u\in\mathbb{T}_n} \mathbf{1}_{\{u=\omega_n^n\}} h\left(\tilde{X}_{u|k}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u|k}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,u|k}(t)\right)\\
= \;& \mathbb{E}_{\mathbb{Q}_\xi} \sum_{u\in\mathbb{T}_n} \mathbb{Q}_\xi(\omega_n^n = u|\mathcal{F}_\infty) h\left(\tilde{X}_{u|k}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u|k}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,u|k}(t)\right)\\
= \;& \mathbb{E}_{\mathbb{Q}_\xi} \sum_{u\in\mathbb{T}_n} \frac{\tilde{X}_u(t)}{W_n(t)} h\left(\tilde{X}_{u|k}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u|k}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,u|k}(t)\right)\\
= \;& \mathbb{E}_{\xi} \sum_{u\in\mathbb{T}_n} \tilde{X}_u(t) h\left(\tilde{X}_{u|k}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u|k}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,u|k}(t)\right)\\
= \;& \mathbb{E}_{\xi} \sum_{u\in\mathbb{T}_k} \tilde{X}_u(t)W_{n-k,u}(t) h\left(\tilde{X}_{u}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,u}(t)\right)\\
= \;& \mathbb{E}_{\xi} \sum_{u\in\mathbb{T}_k} \tilde{X}_u(t) h\left(\tilde{X}_{u}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u}} \tilde{X}_v(t)W_{n-k,v}(t)\right) \mathbb{E}_{T^k\xi} W_{n-k}(t) g\left(W_{n-k}(t)\right).
\end{aligned}$$

□

Remark 3.1. In particular, taking $h = 1$ or $g = 1$ gives

$$\mathbb{E}_{\mathbb{Q}_\xi} g\left(W_{n-k,\omega_k^n}(t)\right) = \mathbb{E}_{T^k\xi} W_{n-k}(t) g\left(W_{n-k}(t)\right), \tag{3.4}$$

$$\mathbb{E}_{\mathbb{Q}_\xi} h\left(\tilde{X}_{\omega_k^n}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right) = \mathbb{E}_{\xi} \sum_{v\in\mathbb{T}_k} \tilde{X}_u(t) h\left(\tilde{X}_{u}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq u}} \tilde{X}_v(t)W_{n-k,v}(t)\right). \tag{3.5}$$

Combing (3.4), (3.5) with (3.3), we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_\xi} h\left(\tilde{X}_{\omega_k^n}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right) g\left(W_{n-k,\omega_k^n}(t)\right)\\
= \;& \mathbb{E}_{\mathbb{Q}_\xi} h\left(\tilde{X}_{\omega_k^n}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right) \mathbb{E}_{\mathbb{Q}_\xi} g\left(W_{n-k,\omega_k^n}(t)\right),
\end{aligned}$$

which means that the random vector $\left(\tilde{X}_{\omega_k^n}(t), \sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right)$ is independent of $W_{n-k,\omega_k^n}(t)$ under $\mathbb{Q}_\xi$. Moreover, for all nonnegative Borel functions f defined on $\mathbb{R}$, by taking $h(x,y) = f(x)$ or $f(y)$, we

obtain

$$\mathbb{E}_{\mathbb{Q}_\xi} f\left(\tilde{X}_{\omega_k^n}(t)\right) = \mathbb{E}_\xi \sum_{u\in\mathbb{T}_k} \tilde{X}_u(t) f\left(\tilde{X}_u(t)\right),$$

$$\mathbb{E}_{\mathbb{Q}_\xi} f\left(\sum_{\substack{v\in\mathbb{T}_k\\ v\neq\omega_k^n}} \tilde{X}_v(t)W_{n-k,v}(t)\right) = \mathbb{E}_\xi \sum_{u\in\mathbb{T}_k} \tilde{X}_u(t) f\left(\sum_{\substack{v\in\mathbb{T}_k\\ v\neq u}} \tilde{X}_v(t)W_{n-k,v}(t)\right).$$

These above assertions generalize the results of Liu ([32], Lemma 4.1) on generalized multiplicative cascades.

3.2 Auxiliary results

In this section, We shall obtain some auxiliary results for the study of the annealed L^p convergence rate of W_n. Let's consider the i.i.d. environment, where (ξ_n) are i.i.d. Denote

$$U_n^{(t)}(s,r) = \mathbb{E}P_n(t)^s W_n(t)^r \quad (s,t\in\mathbb{R}, r>1).$$

We will show two lemmas about $U_n^{(t)}(s,r)$: the first one is a recursive inequality; the second one gives a upper estimation. Particularly, the results for $t=0$ were already shown in [24].

Lemma 3.2. *Let $r>2$. Then*

$$U_n^{(t)}(s,r)^{\frac{1}{r-1}} \le \left[\mathbb{E}m_0(t)^{s-r}m_0(tr)\right]^{\frac{1}{r-1}} U_{n-1}^{(t)}(s,r)^{\frac{1}{r-1}} + \left[\mathbb{E}m_0(t)^s W_1(t)^r\right]^{\frac{1}{r-1}} U_{n-1}^{(t)}(s,r-1)^{\frac{1}{r-1}}. \tag{3.6}$$

Proof. Fix $t\in\mathbb{R}$. Given ξ, we consider the probability $\mathbb{Q}_\xi$ defined in Section 3.1. Notice that

$$W_n(t) = \sum_{u\in\mathbb{T}_n} \tilde{X}_u(t) = \sum_{u\in\mathbb{T}_1} \tilde{X}_u^{(t)} W_{n-1,u}(t).$$

We have

$$\mathbb{E}_\xi W_n(t)^r = \mathbb{E}_{\mathbb{Q}_\xi} W_{n-1}(t)^{r-1} = \mathbb{E}_{\mathbb{Q}_\xi}\left(\sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)W_{n-1,u}(t)\right)^{r-1}.$$

Thus

$$\begin{aligned} U_n^{(t)}(s,r) &= \mathbb{E}P_n(t)^s\mathbb{E}_\xi W_n(t)^r \\ &= \mathbb{E}P_n(t)^s\mathbb{E}_{\mathbb{Q}_\xi}\left(\sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)W_{n-1,u}(t)\right)^{r-1} \\ &= \mathbb{E}_{\mathbb{Q}}\left(P_n(t)^{\frac{s}{r-1}}\sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)W_{n-1,u}(t)\right)^{r-1}, \end{aligned}$$

where the probability $\mathbb{Q}$ is defined as $\mathbb{Q}(B) = \mathbb{E}\mathbb{Q}_\xi(B)$, for any measurable set B.

Fix $n\ge 1$. The set $\mathbb{T}_1$ can be divided into two parts: $\{\omega_1^n\}$ and the set of his brothers $\{u\in\mathbb{T}_1 : u\neq\omega_1^n\}$. We therefore have

$$\begin{aligned} \sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)W_{n-1,u}(t) &= \tilde{X}_{\omega_1^n}(t)W_{n-1,\omega_1^n}(t) + \sum_{\substack{u\in\mathbb{T}_1\\ u\neq\omega_1^n}} \tilde{X}_u(t)W_{n-1,u}(t) \\ &=: M_n + Q_n. \end{aligned}$$

Hence

$$U_n^{(t)}(s,r) = \mathbb{E}_{\mathbb{Q}}\left[P_n(t)^{\frac{s}{r-1}}M_n + P_n(t)^{\frac{s}{r-1}}Q_n\right]^{r-1}.$$

By Minkowski's inequality,

$$U_n^{(t)}(s,r)^{\frac{1}{r-1}} \le \left[\mathbb{E}_{\mathbb{Q}}P_n(t)^s M_n^{r-1}\right]^{\frac{1}{r-1}} + \left[\mathbb{E}_{\mathbb{Q}}P_n(t)^s Q_n^{r-1}\right]^{\frac{1}{r-1}}. \tag{3.7}$$

By Lemma 3.1, we can calculate a.s.,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\xi} M_n^{r-1} &= \mathbb{E}_{\mathbb{Q}_\xi}\left[\tilde{X}_{\omega_1^n}(t) W_{n-1,\omega_1^n}(t)\right]^{r-1} \\
&= \mathbb{E}_\xi \sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)^r \mathbb{E}_{T\xi} W_{n-1}(t)^r \\
&= m_0(tr) m_0(t)^{-r} \mathbb{E}_{T\xi} W_{n-1}(t)^r.
\end{aligned}$$

Therefore, by the independency of (ξ_n),

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} P_n(t)^s M_n^{r-1} &= \mathbb{E} P_n(t)^s \mathbb{E}_{\mathbb{Q}_\xi} M_n^{r-1} \\
&= \mathbb{E} P_n(t)^s m_0(tr) m_0(t)^{-r} \mathbb{E}_{T\xi} W_{n-1}(t)^r \\
&= \mathbb{E} m_0(t)^{s-r} m_0(tr) \mathbb{E} P_{n-1}(t)^s W_{n-1}(t)^r \\
&= \mathbb{E} m_0(t)^{s-r} m_0(tr) U_{n-1}^{(t)}(s,r). \qquad (3.8)
\end{aligned}$$

Similarly, again by Lemma 3.1, we have a.s.

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_\xi} Q_n^{r-1} &= \mathbb{E}_{\mathbb{Q}_\xi}\left(\sum_{\substack{u\in\mathbb{T}_1\\ u\neq\omega_1^n}} \tilde{X}_u(t) W_{n-1,u}(t)\right)^{r-1} \\
&= \mathbb{E}_\xi \sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)\left(\sum_{\substack{v\in\mathbb{T}_1\\ v\neq u}} \tilde{X}_v(t) W_{n-1,v}(t)\right)^{r-1} \\
&\leq \mathbb{E}_\xi \sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)\left(\sum_{\substack{v\in\mathbb{T}_1\\ v\neq u}} \tilde{X}_v(t)\right)^{r-2} \sum_{\substack{v\in\mathbb{T}_1\\ v\neq u}} \tilde{X}_v(t) W_{n-1,v}(t)^{r-1} \\
&= \mathbb{E}_\xi \sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)\left(\sum_{\substack{v\in\mathbb{T}_1\\ v\neq u}} \tilde{X}_v(t)\right)^{r-1} \mathbb{E}_{T\xi} W_{n-1}(t)^{r-1} \\
&\leq \mathbb{E}_\xi \left(\sum_{u\in\mathbb{T}_1} \tilde{X}_u(t)\right)^{r} \mathbb{E}_{T\xi} W_{n-1}(t)^{r-1} \\
&= \mathbb{E}_\xi W_1(t)^r \mathbb{E}_{T\xi} W_{n-1}(t)^{r-1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} P_n(t)^s Q_n^{r-1} &= \mathbb{E} P_n(t)^s \mathbb{E}_{\mathbb{Q}_\xi} Q_n^{r-1} \\
&\leq \mathbb{E} P_n(t)^s \mathbb{E}_\xi W_1(t)^r \mathbb{E}_{T\xi} W_{n-1}(t)^{r-1} \\
&= \mathbb{E} m_0(t)^s W_1(t)^r U_{n-1}^{(t)}(s, r-1). \qquad (3.9)
\end{aligned}$$

Combing (3.8), (3.9) with (3.7), we obtain (3.6). □

Following similar arguments of ([24], Lemma 4.4), we obtain the following lemma which generalize ([24], Lemma 4.4) to BRWRE. The proof is omitted.

Lemma 3.3. *If* $\mathbb{E} m_0(t)^s < \infty$ *and* $\mathbb{E} m_0(t)^s W_1(t)^r < \infty$, *then*

(i) for $r \in (1,2]$,

$$U_n^{(t)}(s,r) \leq Cn\left[\max\left\{\mathbb{E} m_0(t)^{s-r} m_0(tr), \mathbb{E} m_0(t)^s\right\}\right]^n;$$

(ii) for $r \in (b+1, b+2]$, where $b \geq 1$ is an integer,

$$U_n^{(t)}(s,r) \leq Cn^{br}\left[\max\left\{\left(\mathbb{E}m_0(t)^{s-r+i}m_0(t(r-i))\right)_{0\leq i\leq b, i\in\mathbb{N}}, \mathbb{E}m_0(t)^s\right\}\right]^n,$$

where C is a general constant depending on r, s and t.

Lemma 3.4. *The function $f(x) := \mathbb{E}m_0(t)^x m_0(\alpha+\beta x)$ (t, α and $\beta \in \mathbb{R}$ are fixed) is* log *convex.*

Proof. For $\lambda \in (0,1)$, $\forall x_1, x_2$, using Hölder's inequality, we have

$$m_0(\alpha + \beta(\lambda x_1 + (1-\lambda)x_2)) \leq m_0(\alpha+\beta x_1)^\lambda m_0(\alpha+\beta x_2)^{1-\lambda},$$

which means that $m_0(\alpha+\beta x)$ is log convex. Noticing the inequality above and using Hölder's inequality again, we get

$$\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) &= \mathbb{E}m_0(t)^{\lambda x_1+(1-\lambda)x_2} m_0(\alpha+\beta(\lambda x_1+(1-\lambda)x_2)) \\
&\leq \mathbb{E}\left[m_0(t)^{x_1}m_0(\alpha+\beta x_1)\right]^\lambda \left[m_0(t)^{x_2}m_0(\alpha+\beta x_2)\right]^{1-\lambda} \\
&\leq \left[\mathbb{E}m_0(t)^{x_1}m_0(\alpha+\beta x_1)\right]^\lambda \left[\mathbb{E}m_0(t)^{x_2}m_0(\alpha+\beta x_2)\right]^{1-\lambda} \\
&= f(x_1)^\lambda f(x_2)^{1-\lambda},
\end{aligned}$$

which confirms the log-convexity of f. □

3.3 Proofs of Theorems 1.2 and 1.3

Recall the martingale $\hat{A}_n$ introduced in Section 2. Similar to the quenched case, we need to study the p-th ($p>1$) moment of $\hat{A}_n$ under annealed law $\mathbb{P}$. We distinguish two case: $1<p<2$ and $p \geq 2$.

Proposition 3.5 (Annealed moments of $\hat{A}_n$: case $1<p<2$)**.** *Assume (1.14) and that (ξ_n) are i.i.d.. Let $1<p<2$ and $\rho \geq 1$. If $\mathbb{E}[\mathbb{E}_\xi W_1^r]^{p/r} < \infty$ and $\rho\left[\mathbb{E}m_0(r)^{p/r}\right]^{1/p} < 1$ for some $r \in [p,2]$, then $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$.*

Proof. Similar to the proof of Theorem 2.3, appling Burkholder's inequality to $\hat{A}_n$ under annealed law $\mathbb{P}$ and noticing (2.5), we have

$$\begin{aligned}
\sup_n \mathbb{E}|\hat{A}_n|^p &\leq C\sum_{n=0}^\infty \rho^{pn}\mathbb{E}\left[P_n(r)\mathbb{E}_{T^n\xi}|W_1-1|^r\right]^{p/r} \\
&= C\mathbb{E}(\mathbb{E}_\xi|W_1-1|^r)^{p/r}\sum_{n=0}^\infty \rho^{pn}[\mathbb{E}m_0(r)^{p/r}]^n.
\end{aligned}$$

Thus $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ if $\mathbb{E}[\mathbb{E}_\xi W_1^r]^{p/r} < \infty$ and $\rho(\mathbb{E}m_0(r)^{p/r})^{1/p} < 1$. □

Now we consider the case where $p \geq 2$. The proposition below gives a sufficient and necessary condition for the existence of uniform p-th moment of $\hat{A}_n$ under annealed law $\mathbb{P}$.

Proposition 3.6 (Annealed moments of $\hat{A}_n$: case $p \geq 2$)**.** *Assume (1.14), (1.15) and that (ξ_n) are i.i.d.. Let $p \geq 2$ and $\rho \geq 1$. Then $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$ if and only if $\mathbb{E}W_1^p < \infty$ and $\rho\max\{[\mathbb{E}m_0(p)]^{1/p}, [\mathbb{E}m_0(2)^{p/2}]^{1/p}\} < 1$.*

Proof. (i) The necessity. Since $\sup_n \mathbb{E}|\hat{A}_n|^p < \infty$, we have $\mathbb{E}W_1^p < \infty$. Furthermore, by Burkholder's inequality, we can calculate for all $r \in [2,p]$,

$$\begin{aligned}
\sup_n \mathbb{E}|\hat{A}_n|^p &\geq C\sum_{n=0}^\infty \rho^{pn}\mathbb{E}\left(\sum_{u\in\mathbb{T}_n}\tilde{X}_u^2(W_{1,u}-1)^2\right)^{p/2} \\
&\geq C\sum_{n=0}^\infty \rho^{pn}\mathbb{E}\left(\mathbb{E}_\xi\sum_{u\in\mathbb{T}_n}\tilde{X}_u^r|W_{1,u}-1|^r\right)^{p/r} \\
&= C\mathbb{E}(\mathbb{E}_\xi|W_1-1|^r)^{p/r}\sum_{n=0}^\infty \rho^{pn}[\mathbb{E}m_0(r)^{p/r}]^n.
\end{aligned}$$

Therefore the series $\sum_n \rho^{pn}[\mathbb{E}m_0(r)^{p/r}]^n < \infty$ for all $r \in [2,p]$, which implies that $\rho[\mathbb{E}m_0(r)^{p/r}]^{1/p} < 1$ for all $r \in [2,p]$. Taking $r = 2, p$, we get $\rho \max\{[\mathbb{E}m_0(p)]^{1/p}, [\mathbb{E}m_0(2)^{p/2}]^{1/p}\} < 1$.

(ii) The sufficiency. By Burkholder's inequality and Minkowski's inequality,

$$\begin{aligned}\sup_n \mathbb{E}|\hat{A}_n|^p &\leq C\mathbb{E}\left(\sum_{n=0}^{\infty} \rho^{2n}(W_{n+1}-W_n)^2\right)^{p/2} \\ &\leq C\left(\sum_{n=0}^{\infty} \rho^{2n}(\mathbb{E}|W_{n+1}-W_n|^p)^{2/p}\right)^{p/2}.\end{aligned}$$

By (2.8),

$$\mathbb{E}|W_{n+1}-W_n|^p \leq C\mathbb{E}|W_1-1|^p\mathbb{E}P_n(2)^{p/2}W_n(2)^{p/2} = C\mathbb{E}|W_1-1|^p U_n^{(2)}(p/2,p/2). \qquad (3.10)$$

Since $\mathbb{E}m_0(2)^{p/2} < \infty$, and

$$\mathbb{E}m_0(2)^{p/2}W_1(2)^{p/2} = \mathbb{E}\left(\sum_{u\in\mathbb{T}_1} \tilde{X}_u^2\right)^{p/2} \leq \mathbb{E}\left(\sum_{u\in\mathbb{T}_1} \tilde{X}_u\right)^p = \mathbb{E}W_1^p < \infty,$$

by Lemma 3.3,

$$U_n^{(2)}(p/2,p/2) \leq Cn^\gamma \left[\max\left\{\left(\mathbb{E}m_0(2)^i m_0(p-2i)\right)_{0\leq i\leq b, i\in\mathbb{N}}, \mathbb{E}m_0(2)^{p/2}\right\}\right]^n \qquad (3.11)$$

for $p/2 \in (b+1, b+2]$ ($b \geq 0$ is an integer), where $\gamma = 1$ for $b = 0$ and $\gamma = bp/2$ for $b \geq 1$. Observing that $\mathbb{E}m_0(2)^x m_0(p-2x)$ is log convex (see Lemma 3.4), we have

$$\begin{aligned}&\max\left\{\left(\mathbb{E}m_0(2)^i m_0(p-2i)\right)_{0\leq i\leq b, i\in\mathbb{N}}, \mathbb{E}m_0(2)^{p/2}\right\} \\ \leq\ &\sup_{0\leq x\leq p/2-1}\{\mathbb{E}m_0(2)^x m_0(p-2x)\} = \max\{\mathbb{E}m_0(p), \mathbb{E}m_0(2)^{p/2}\}. \qquad (3.12)\end{aligned}$$

Thus, by (3.10), (3.11) and (3.12), we have

$$\mathbb{E}|W_{n+1}-W_n|^p \leq Cn^\gamma\left[\max\left\{\mathbb{E}m_0(p), \mathbb{E}m_0(2)^{p/2}\right\}\right]^n.$$

Hence we obtain

$$\sum_n \rho^{2n}(\mathbb{E}|W_{n+1}-W_n|^p)^{2/p} \leq C\sum_n \rho^{2n}n^{2\gamma/p}\left[\max\left\{[\mathbb{E}m_0(p)]^{2/p}, [\mathbb{E}m_0(2)^{p/2}]^{2/p}\right\}\right]^n.$$

The right side is finite if and only if $\rho\max\{[\mathbb{E}m_0(p)]^{1/p}, [\mathbb{E}m_0(2)^{p/2}]^{1/p}\} < 1$. □

Now we prove Theorems 1.2 and 1.3, using the moment results of $\hat{A}_n$.

Proof of Theorem 1.2. (i) The sufficiency. For $1 < p < 2$, applying Proposition 3.5 with $\rho = 1$, we obtain $\sup_n \mathbb{E}W_n^p < \infty$, which is equivalent to $W_n \to W$ in $\mathbb{P}$-L^p. For $p \geq 2$, by the log convexity of $m_0(x)$ and Jensen's inequality, one has

$$\mathbb{E}m_0(2)^{p/2} \leq \mathbb{E}m_0(p)^{p/2(p-1)} \leq [\mathbb{E}m_0(p)]^{p/2(p-1)}.$$

So the condition $\mathbb{E}m_0(p) < 1$ ensures $\max\{[\mathbb{E}m_0(p)]^{1/p}, [\mathbb{E}m_0(2)^{p/2}]^{1/p}\} < 1$. Applying Proposition 3.6 with $\rho = 1$ yields the results.

(ii) The necessity. Notice that $\sup_n \mathbb{E}W_n^p < \infty$ ensures that $\mathbb{E}W_1^p < \infty$ and $0 < \mathbb{E}W^p < \infty$. One can see that W satisfies the equation

$$W = \sum_{u\in\mathbb{T}_1} \tilde{X}_u W_{(u)} \qquad a.s.,$$

where $W_{(u)}$ denotes the limit random variable of the martingale $W_{n,u}$, and the distribution of $W_{(u)}$ is $\mathbb{P}_\xi(W_{(u)} \in \cdot) = \mathbb{P}_{T^{|u|}\xi}(W \in \cdot)$. We have

$$W^p = \left(\sum_{u \in \mathbb{T}_1} \tilde{X}_u W_{(u)} \right)^p \geq \sum_{u \in \mathbb{T}_1} \tilde{X}_u^p W_{(u)}^p \quad a.s.,$$

and the strict inequality holds with positive probability. Thus

$$\mathbb{E}W^p > \mathbb{E} \sum_{u \in \mathbb{T}_1} \tilde{X}_u^p W_{(u)}^p = \mathbb{E}m_0(p)\mathbb{E}W^p,$$

which implies that $\mathbb{E}m_0(p) < 1$.

□

Proof of Theorem 1.3. The assertion (a) is consequently from Proposition 3.5 with $\rho > 1$ and the assertion (b) is from Proposition 3.6 with $\rho > 1$. □

4 Uniform convergence; Proof of Theorem 1.4

In this section, we study the uniform convergence of the martingale $W_n(t)$, regarding $W_n(t)$ as the function of t (so t is not fixed). Here we just consider the quenched uniform convergence and give the proof of Theorem 1.4. The annealed uniform convergence can be obtain almost in the same way, so we omit the proof of Theorem 1.5. The basic tool is still the inequalities for martingale. But in contrast to the convergence for t fixed, we should consider the superior on a interval of t while estimating the moment of $W_n(t)$. We first provide two related lemmas.

Lemma 4.1. *Let $D = [t_1, t_2] \subset I$. If $D \subset [0, \infty)$ or $D \subset (-\infty, 0]$, then there exists a constant $p_D > 1$ such that for any $p \in (1, p_D]$,*

$$\mathbb{E} \log \sup_{t \in D} \left[\frac{m_0(pt)}{m_0(t)^p} \right] < 0.$$

Proof. For $t_0 \in \mathbb{R}$, set $g_{t_0}(p) = \Lambda(pt_0) - p\Lambda(t_0)$, whose derivative is

$$g_{t_0}'(p) = t_0 \Lambda'(pt_0) - \Lambda(t_0).$$

If $t_0 \in I$, then $g_{t_0}'(1) = t_0\Lambda'(t_0) - \Lambda(t_0) < 0$. Hence there exists a $p_0 > 1$ such that $g_{t_0}(p)$ is strictly decreasing on $(1, p_0]$, so that $g_{t_0}(p) < g_{t_0}(1) = 0$ for all $p \in (1, p_0]$.

Assume $D \subset [0, \infty)$. Since $t_2 \in I$, there exists $p_2 > 1$ such that $g_{t_2}(p) < 0$ for all $p \in (1, p_2]$. For $p > 1$ fixed, set $f_p(t) = \log m_0(pt) - p \log m_0(t)$. Clearly, $\mathbb{E}f_p(t) = g_t(p)$. The derivative of $f_p(t)$ is

$$f_p'(t) = p \left(\frac{m_0'(pt)}{m_0(pt)} - \frac{m_0'(t)}{m_0(t)} \right).$$

By the convexity of $\log m_0(t)$, we see that the function $\frac{m_0'(t)}{m_0(t)}$ is increasing, so that $f_p'(t) \geq 0$ for $t \geq 0$ and $f_p'(t) \leq 0$ for $t \leq 0$. Thus $f_p(t)$ is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. Since $D \subset [0, \infty)$, we have

$$\sup_{t \in D} f_p(t) = f_p(t_2).$$

Take $p_D = p_2$. Then for any $p \in (1, p_D]$,

$$\mathbb{E} \log \sup_{t \in D} \left[\frac{m_0(pt)}{m_0(t)^p} \right] = \mathbb{E} \sup_{t \in D} \log \left[\frac{m_0(pt)}{m_0(t)^p} \right] = \mathbb{E} \sup_{t \in D} f_p(t) = \mathbb{E}f_p(t_2) = g_{t_2}(p) < 0.$$

For $D \subset (-\infty, 0]$, the proof is similar. □

Remark 4.1. Set $G_{t_0}(p) = \mathbb{E}\left[\frac{m_0(pt)}{m_0(t)^p}\right]$ for $t_0 \in \mathbb{R}$ fixed. If $t_0 \in I'$, then there exists a $p_0 > 1$ such tat $G_{t_0}(p_0) < 1$. The log convexity of $G_{t_0}(p)$ yields $G_{t_0}(p) < 1$ for any $p \in (1, p_0]$. Notice that $G_t(p) = \mathbb{E}e^{f_p(t)}$.

With similar arguments to the proof of Lemma 4.1, we can obtain that if $D=[t_1,t_2]\subset I'$, then there exists a constant $p_D>1$ such that for any $p\in(1,p_D]$,

$$\sup_{t\in D}\mathbb{E}\left[\frac{m_0(pt)}{m_0(t)^p}\right]<1.$$

This result could be used to study of the annealed uniform convergence in the role of replacing Lemma 4.1 for quenched case.

Recall that $\underline{m}_0=\inf_{t\in I}m_0(t)$ and $\underline{m}=essinf\underline{m}_0$.

Lemma 4.2. *Let $D=[t_1,t_2]\subset I\bigcap\Omega_1$. If $\mathbb{E}\log^-\underline{m}_0<\infty$, then there exists a constant $p_D>1$ such that*

$$\mathbb{E}\log\sup_{t\in D}\mathbb{E}_\xi W_1(t)^{p_D}<\infty.$$

Proof. Since $D\subset I$, we have $\inf_{t\in D}m_0(t)\geq\underline{m}_0$. Notice that for all $t\in D$,

$$\begin{aligned}
W_1(t)=\sum_{u\in\mathbb{T}_1}\frac{e^{tS_u}}{m_0(t)} &\leq \frac{1}{\underline{m}_0}\left(\sum_{u\in\mathbb{T}_1}e^{t_2S_u}\mathbf{1}_{\{S_u\geq0\}}+\sum_{u\in\mathbb{T}_1}e^{t_1S_u}\mathbf{1}_{\{S_u<0\}}\right)\\
&\leq \frac{1}{\underline{m}_0}\left(\tilde{Z}_1(t_1)+\tilde{Z}_1(t_2)\right) \qquad (4.1)
\end{aligned}$$

Thus $\mathbb{E}\log\sup_{t\in D}\mathbb{E}_\xi W_1(t)^p<\infty$ if $\mathbb{E}\log^-\underline{m}_0<\infty$ and $\mathbb{E}\log^+\mathbb{E}\tilde{Z}_1(t_i)^p<\infty$ $(i=1,2)$. Since $t_i\in\Omega_1$, there exists $p_i>1$ such that $\mathbb{E}\log\mathbb{E}_\xi W_1(t_i)^{p_i}<\infty$, which is equivalent to $\mathbb{E}\log^+\mathbb{E}_\xi\tilde{Z}_1(t_i)^{p_i}<\infty$ under condition $\mathbb{E}|\log m_0(t_i)|<\infty$. Taking $p_D=\min\{p_1,p_2\}$ completes the proof. □

Lemma 4.3. *Let $D=[t_1,t_2]\subset I\bigcap\Omega_2$ and $W_D^*=\sup_{t\in D}W_1(t)$. If $\underline{m}>0$, then $\mathbb{E}W_D^*\log^+W_D^*<\infty$.*

Proof. By (4.1), we see that

$$W_D^*\leq\frac{1}{\underline{m}}\left(\tilde{Z}_1(t_1)+\tilde{Z}_1(t_2)\right).$$

Since $t_i\in\Omega_2$ $(i=1,2)$, we have $\mathbb{E}\tilde{Z}_1(t_i)\log^+\tilde{Z}_1(t_i)<\infty$, which ensures $\mathbb{E}W_D^*\log^+W_D^*<\infty$. □

Now we prove Theorem 1.4.

Proof of Theorem 1.4. We first consider the assertion (a). Clearly, it suffices to prove that for each $t_0\in I\bigcap\Omega_1$, there exists an interval $D=[t_0-\varepsilon,t_0+\varepsilon]\subset I\bigcap\Omega_1$ ($\varepsilon>0$ small enough) such that the series

$$\sum_n\sup_{t\in D}\left(\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|^p\right)^{1/p}<\infty \qquad a.s. \qquad (4.2)$$

for suitable $1<p\leq2$. By (2.5), we have a.s.,

$$\begin{aligned}
\sup_{t\in D}\left(\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|^p\right)^{1/p} &\leq C\frac{P_n(pt)^{1/p}}{P_n(t)}\left(\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\right)^{1/p}\\
&\leq C\left(\sup_{t\in D}\left[\frac{P_n(pt)}{P_n(t)^p}\right]\right)^{1/p}\left(\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\right)^{1/p}. \qquad (4.3)
\end{aligned}$$

Here and after the general constant C does not depend on t. Decompose $D=D^+\bigcup D^-$, where $D^+=D\bigcap[0,\infty)$ and $D^-=(-\infty,0]$. Then

$$\begin{aligned}
\sup_{t\in D}\left[\frac{P_n(pt)}{P_n(t)^p}\right] &\leq \max\left\{\sup_{t\in D^+}\left[\frac{P_n(pt)}{P_n(t)^p}\right],\sup_{t\in D^-}\left[\frac{P_n(pt)}{P_n(t)^p}\right]\right\}\\
&\leq \max\left\{\prod_{i=0}^{n-1}\sup_{t\in D^+}\left[\frac{m_i(pt)}{m_i(t)^p}\right],\prod_{i=0}^{n-1}\sup_{t\in D^-}\left[\frac{m_i(pt)}{m_i(t)^p}\right]\right\} \qquad (4.4)
\end{aligned}$$

By ergodic theorem and Lemma 4.1, a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\log\left(\prod_{i=0}^{n-1}\sup_{t\in D^+}\left[\frac{m_i(pt)}{m_i(t)^p}\right]\right)=\mathbb{E}\log\sup_{t\in D^+}\left[\frac{m_0(pt)}{m_0(t)^p}\right]<0 \tag{4.5}$$

for suitable $1<p\leq 2$. The same is true with D^+ replaced by D^-. Thus, there exists a constant $a_D>1$ such that

$$\max\left\{\prod_{i=0}^{n-1}\sup_{t\in D^+}\left[\frac{m_i(pt)}{m_i(t)^p}\right],\prod_{i=0}^{n-1}\sup_{t\in D^-}\left[\frac{m_i(pt)}{m_i(t)^p}\right]\right\}<a_D^{-n}\qquad a.s. \tag{4.6}$$

for n large enough. Combing (4.4) and (4.7) yields

$$\sup_{t\in D}\left[\frac{P_n(pt)}{P_n(t)^p}\right]<a_D^{-n}\qquad a.s. \tag{4.7}$$

for n large enough. Hence the a.s. convergence of the series

$$\sum_n a_D^{-n}\left(\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\right)^{1/p} \tag{4.8}$$

implies (4.2). Since $a_D>1$ and $\mathbb{E}\log\sup\limits_{t\in D}\mathbb{E}_\xi W_1(t)^p<\infty$, the a.s. convergence of (4.8) is ensured by Lemma 2.1.

We next prove the assertion (b). For $t_0\in I\bigcap\Omega_2$, take $\varepsilon>0$ small enough such that the series

$$\sum_n\sup_{t\in D}\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|<\infty\qquad a.s.. \tag{4.9}$$

We will use a truncation method, similarly to Biggins [10]. Set $I_n=\mathbf{1}_{\{|W_1(t)-1|\geq c^n\}}$ and $\bar{I}_n=1-I_n$, where $c>1$ is a constant whose value will be taken later. Using ([10], Lemma 4), we get for $1\leq p\leq 2$,

$$\mathbb{E}_\xi|W_{n+1}(t)-W_n(t)|\leq C\left(\mathbb{E}_{T^n\xi}|W_1(t)-1|I_n+\left(\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\bar{I}_n\right)^{1/p}\left[\frac{P_n(pt)}{P_n(t)^p}\right]^{1/p}\right).$$

To get (4.9), we need to consider the a.s. convergence of the two series:

$$\sum_n\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|I_n, \tag{4.10}$$

$$\sum_n\left(\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\bar{I}_n\right)^{1/p}\left(\sup_{t\in D}\left[\frac{P_n(pt)}{P_n(t)^p}\right]\right)^{1/p}. \tag{4.11}$$

For (4.10), observe that

$$\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|I_n\leq\mathbb{E}_{T^n\xi}(W_D^*+1)\mathbf{1}_{\{W_D^*+1\geq c^n\}}.$$

By Lemma (4.3), we see $\mathbb{E}W_D^*\log^+W_D^*<\infty$. Thus,

$$\begin{aligned}\mathbb{E}\left(\sum_n\mathbb{E}_{T^n\xi}(W_D^*+1)\mathbf{1}_{\{W_D^*+1\geq c^n\}}\right) &= \sum_n\mathbb{E}(W_D^*+1)\mathbf{1}_{\{W_D^*+1\geq c^n\}}\\ &\leq C\mathbb{E}(W_D^*+1)\log(W_D^*+1)<\infty,\end{aligned}$$

which leads to the a.s. convergence of (4.10). For (4.11), Notice (4.8) and the fact that

$$\sup_{t\in D}\mathbb{E}_{T^n\xi}|W_1(t)-1|^p\bar{I}_n\leq c^{np}.$$

Taking $1<c<a_D$ yields the a.s. convergence of (4.10). The proof is completed. ◻

5 Moderate deviation principles

5.1 Moderate deviation principle for $\frac{\mathbb{E}_\xi Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$; Proof of Theorem 1.6

We first study the moderate deviations for the quenched means.

Theorem 5.1 (Moderate deviation principle for quenched means $\frac{\mathbb{E}_\xi Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$)**.** *Write $\pi_0 = m_0(0)$. If $\|\frac{1}{\pi_0}\mathbb{E}_\xi \sum_{i=1}^N e^{\delta|L_i|}\|_\infty := esssup\frac{1}{\pi_0}\mathbb{E}_\xi \sum_{i=1}^N e^{\delta|L_i|} < \infty$ for some $\delta > 0$ and $\mathbb{E}_\xi \sum_{i=1}^N L_i = 0$ a.s., then the sequence of probability measures $A \mapsto \frac{\mathbb{E}_\xi Z_n(a_n A)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ satisfies a principle of moderate deviation: for each measurable subset A of $\mathbb{R}$,*

$$\begin{aligned}
-\frac{1}{2\sigma^2}\inf_{x\in A^o} x^2 &\leq \liminf_{n\to\infty}\frac{n}{a_n^2}\log\frac{\mathbb{E}_\xi Z_n(a_nA)}{\mathbb{E}_\xi Z_n} \\
&\leq \limsup_{n\to\infty}\frac{n}{a_n^2}\log\frac{\mathbb{E}_\xi Z_n(a_nA)}{\mathbb{E}_\xi Z_n} \leq -\frac{1}{2\sigma^2}\inf_{x\in \bar{A}} x^2 \qquad a.s., \qquad (5.1)
\end{aligned}$$

where $\sigma^2 = \mathbb{E}\frac{1}{\pi_0}\sum_{i=1}^N L_i^2$, and A^o denotes the interior of A and $\bar{A}$ its closure.

Proof. Consider the probability measures $q_n(\cdot) = \frac{\mathbb{E}_\xi Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$. Let

$$\lambda_n(t) = \log\int e^{tx} q_n(dx) = \log\left[\frac{\mathbb{E}_\xi \int e^{a_n^{-1}tx} Z_n(dx)}{\mathbb{E}_\xi Z_n(\mathbb{R})}\right].$$

Then

$$\lambda_n(t) = \log\left[\frac{P_n(a_n^{-1}t)}{P_n(0)}\right] = \sum_{i=0}^{n-1}\log\left[\frac{m_i(a_n^{-1}t)}{\pi_i}\right]. \qquad (5.2)$$

Set $\lambda(t) = \frac{1}{2}\sigma^2 t^2$, whose Legendre-transform is $\lambda^*(x) = \sup_{x\in\mathbb{R}}\{tx - \lambda(t)\} = \frac{x^2}{2\sigma^2}$. We shall show that for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty}\frac{n}{a_n^2}\lambda_n(\frac{a_n^2}{n}t) = \lambda(t) \qquad a.s.. \qquad (5.3)$$

Then (5.3) a.s. holds for all rational t, and hence for all $t \in \mathbb{R}$ by the convexity of $\lambda_n(t)$ and the continuity of $\lambda(t)$. By the Gärtner-Ellis theorem (cf. [17], p52, Exercises 2.3.20), we get (5.1).

Put $\Delta_{n,i} = \frac{m_i(\frac{a_n}{n}t)}{\pi_i} - 1$. We will see that for each $t \in \mathbb{R}$,

$$\sup_i |\Delta_{n,i}| < 1 \qquad a.s. \qquad (5.4)$$

for n large enough. Denote $M = \|\frac{1}{\pi_0}\mathbb{E}_\xi\sum_{i=1}^N e^{\delta|L_i|}\|_\infty$. Then $\sup_n \frac{1}{\pi_n}\mathbb{E}_\xi\int e^{\delta|x|}X_n(dx) \leq M$ a.s., so that for n large enough,

$$\sum_{k=0}^\infty \frac{1}{\pi_i}\mathbb{E}_\xi\int\frac{1}{k!}\left|\frac{a_n}{n}tx\right|^k X_i(dx) \leq M \qquad a.s..$$

Notice that $\frac{1}{\pi_n}\mathbb{E}_\xi\int xX_n(dx) = \frac{1}{\pi_n}\mathbb{E}_\xi\sum_{i=1}^N L_i = 0$ a.s.. Therefore a.s.,

$$\begin{aligned}
\Delta_{n,i} &= \frac{1}{\pi_i}\mathbb{E}_\xi\int e^{a_n n^{-1}tx}X_i(dx) - 1 \\
&= \frac{1}{\pi_i}\mathbb{E}_\xi\int\left(\sum_{k=0}^\infty\frac{1}{k!}\left(\frac{a_n}{n}tx\right)^k\right)X_i(dx) - 1 \\
&= \sum_{k=0}^\infty\frac{1}{k!}\left(\frac{a_n}{n}t\right)^k\frac{1}{\pi_i}\mathbb{E}_\xi\int x^k X_i(dx) - 1 \\
&= \sum_{k=2}^\infty\frac{1}{k!}\left(\frac{a_n}{n}t\right)^k\frac{1}{\pi_i}\mathbb{E}_\xi\int x^k X_i(dx). \qquad (5.5)
\end{aligned}$$

It follows that

$$\begin{aligned}
\sup_i |\Delta_{n,i}| &\leq \sup_i \left[\sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{a_n}{n} |t| \right)^k \frac{1}{\pi_i} \mathbb{E}_\xi \int |x|^k X_i(dx) \right] \\
&\leq \sup_i \left[\sum_{k=2}^{\infty} \left(\frac{a_n}{n} \frac{|t|}{\delta} \right)^k \frac{1}{\pi_i} \mathbb{E}_\xi \int e^{\delta |x|} X_i(dx) \right] \\
&\leq M \sum_{k=2}^{\infty} \left(\frac{a_n}{n} \frac{|t|}{\delta} \right)^k \leq M_1 \left(\frac{a_n}{n} \right)^2 \qquad a.s., \qquad (5.6)
\end{aligned}$$

where $M_1 > 0$ is a constant (it depends on t). Hence (5.4) holds for n large enough.

Now we calculate the limit (5.3). By (5.2) and (5.4), we have for n large enough, a.s.,

$$\begin{aligned}
\frac{n}{a_n^2} \lambda_n \left(\frac{a_n^2}{n} t \right) &= \frac{n}{a_n^2} \sum_{i=0}^{n-1} \log (1 + \Delta_{n,i}) \\
&= \frac{n}{a_n^2} \sum_{i=0}^{n-1} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (\Delta_{n,i})^j \\
&= \frac{n}{a_n^2} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{i=0}^{n-1} (\Delta_{n,i})^j \\
&= \frac{n}{a_n^2} \sum_{i=0}^{n-1} \Delta_{n,i} + \frac{n}{a_n^2} \sum_{j=2}^{\infty} \frac{(-1)^{j+1}}{j} \sum_{i=0}^{n-1} (\Delta_{n,i})^j \\
&= : A_n + B_n.
\end{aligned}$$

For B_n, by (5.6),

$$|B_n| \leq \frac{n}{a_n^2} \sum_{j=2}^{\infty} \frac{1}{j} \sum_{i=0}^{n-1} |\Delta_{n,i}|^j \leq \sum_{j=2}^{\infty} M_1^j \left(\frac{a_n}{n} \right)^{2j-2} \leq M_2 \left(\frac{a_n}{n} \right)^2 \to 0 \quad a.s. \quad \text{as } n \to \infty,$$

where $M_2 > 0$ is a constant. For A_n, by (5.5), a.s.,

$$\begin{aligned}
A_n &= \frac{n}{a_n^2} \sum_{i=0}^{n-1} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{a_n}{n} t \right)^k \frac{1}{\pi_i} \mathbb{E}_\xi \int x^k X_i(dx) \\
&= \frac{n}{a_n^2} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{a_n}{n} t \right)^k \sum_{i=0}^{n-1} \frac{1}{\pi_i} \mathbb{E}_\xi \int x^k X_i(dx) \\
&= \frac{n}{a_n^2} \frac{1}{2} \left(\frac{a_n}{n} t \right)^2 \sum_{i=0}^{n-1} \frac{1}{\pi_i} \mathbb{E}_\xi \int x^2 X_i(dx) \\
&\quad + \frac{n}{a_n^2} \sum_{k=3}^{\infty} \frac{1}{k!} \left(\frac{a_n}{n} t \right)^k \sum_{i=0}^{n-1} \frac{1}{\pi_i} \mathbb{E}_\xi \int x^k X_i(dx) \\
&= : C_n + D_n.
\end{aligned}$$

The ergodic theorem gives

$$\lim_{n \to \infty} C_n = \frac{1}{2} t^2 \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{\pi_i} \mathbb{E}_\xi \int x^2 X_i(dx) = \frac{1}{2} \sigma^2 t^2 = \lambda(t) \quad a.s..$$

To get (5.3), it remains to show that D_n is negligible. Clearly, a.s.

$$\begin{aligned}
|D_n| &\leq \frac{n}{a_n^2} \sum_{k=3}^{\infty} \frac{1}{k!} \left(\frac{a_n}{n} \frac{|t|}{\delta} \right)^k \sum_{i=0}^{n-1} \frac{1}{\pi_i} \mathbb{E}_\xi \int |\delta x|^k X_i(dx) \\
&\leq M \sum_{k=3}^{\infty} \left(\frac{a_n}{n} \frac{|t|}{\delta} \right)^{k-2} \leq M_3 \frac{a_n}{n} \to 0 \qquad a.s. \qquad \text{as } n \to \infty,
\end{aligned}$$

where $M_3 > 0$ is a constant. This completes the proof. $\square$

The moderate deviation principle for $\frac{Z_n(a_n\cdot)}{Z_n(\mathbb{R})}$ comes from Theorem 5.1 and the uniform convergence of $W_n(t)$ (Theorem 1.4).

Proof of Theorem 1.6. Let

$$\Gamma_n(t) = \log\left[\frac{\int e^{a_n^{-1}tx} Z_n(dx)}{Z_n(\mathbb{R})}\right] = \log\left[\frac{\tilde{Z}_n(a_n^{-1}t)}{Z_n(\mathbb{R})}\right].$$

Notice that

$$\frac{n}{a_n^2}\Gamma_n(\frac{a_n^2}{n}t) = \frac{n}{a_n^2}\log W_n(\frac{a_n}{n}t) + \frac{n}{a_n^2}\lambda_n(\frac{a_n^2}{n}t) - \frac{n}{a_n^2}\log W_n(0). \tag{5.7}$$

As $m_0'(0) = \mathbb{E}_\xi \sum_{i=1}^{N} L_i = 0$, the log convexity of $m_0(x)$ gives $\underline{m}_0 = \pi_0$. Since $\mathbb{E}|\log \pi_0| < \infty$ and $0 \in I \bigcap \Omega_1$, by Theorem 1.4, $W_n(t)$ converges uniformly a.s. to $W(t)$ on $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, so that $W_n(t)$ is continuous at 0. Thus

$$W_n(\frac{a_n}{n}t) \to W(0) \qquad a.s. \quad \text{as } n \to \infty.$$

The assumption (1.10) implies that $W(0) > 0$ a.s. on $\{Z_n(\mathbb{R}) \to \infty\}$. Letting $n \to \infty$ and using (5.3), we obtain for each $t \in \mathbb{R}$,

$$\lim_{n\to} \frac{n}{a_n^2}\Gamma_n(\frac{a_n^2}{n}t) = \lambda(t) = \frac{1}{2}\sigma^2 t^2 \quad \text{a.s. on } \{Z_n(\mathbb{R}) \to \infty\}. \tag{5.8}$$

So (5.8) a.s. holds for all rational t, and therefore for all $t \in \mathbb{R}$ by the convexity of $\Gamma_n(t)$ and the continuity of $\lambda(t)$. Then apply the Gärtner-Ellis theorem.

$\square$

5.2 Moderate deviation principles for $\mathbb{E}\frac{Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ and $\frac{\mathbb{E}Z_n(a_n\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$

Finally, in i.i.d environment, we also have moderate deviation principles for $\mathbb{E}\frac{Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$ and $\frac{\mathbb{E}Z_n(a_n\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$.

Theorem 5.2 (Moderate deviation principle for $\mathbb{E}\frac{Z_n(a_n\cdot)}{\mathbb{E}_\xi Z_n(\mathbb{R})}$)**.** *Assume that ξ_n are i.i.d.. Write $\pi_0 = m_0(0)$. If $\mathbb{E}\frac{1}{\pi_0}\sum_{i=1}^{N} e^{\delta|L_i|} < \infty$ for some $\delta < 0$ and $\mathbb{E}\frac{1}{\pi_0}\sum_{i=1}^{N} L_i = 0$, then the sequence of finite measures $A \mapsto \mathbb{E}\frac{Z_n(a_nA)}{\mathbb{E}Z_n(\mathbb{R})}$ satisfies a principle of moderate deviation: for each measurable subset A of $\mathbb{R}$,*

$$\begin{aligned}
-\frac{1}{2\sigma^2}\inf_{x\in A^o} x^2 \quad &\le \quad \liminf_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{E}\frac{Z_n(a_nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \\
&\le \quad \limsup_{n\to\infty} \frac{n}{a_n^2}\log \mathbb{E}\frac{Z_n(a_nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \le -\frac{1}{2\sigma^2}\inf_{x\in \bar{A}} x^2,
\end{aligned} \tag{5.9}$$

where $\sigma^2 = \mathbb{E}\frac{1}{\pi_0}\sum_{i=1}^{N} L_i^2$, and A^o denotes the interior of A and $\bar{A}$ its closure.

Proof. Let

$$\lambda_n(t) = \log \mathbb{E}\frac{\int e^{a_n^{-1}tx} Z_n(dx)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \quad \text{and} \quad \lambda(t) = \frac{1}{2}\sigma^2 t^2.$$

It suffices to show that for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{n}{a_n^2}\lambda_n(\frac{a_n^2}{n}t) = \lambda(t).$$

Then (5.9) holds by the Gärtner-Ellis theorem. In fact, the condition $\mathbb{E}\frac{1}{\pi_0}\sum_{i=1}^{N} e^{\delta|L_i|} < \infty$ gives

$$\mathbb{E}\frac{1}{\pi_0}\int e^{a_n n^{-1}|tx|} X_0(dx) < \infty \quad \text{for } n \text{ large enough.}$$

Thus

$$\begin{aligned}
\lambda_n(\frac{a_n^2}{n}t) &= n\log\mathbb{E}\left[\frac{m_0(\frac{a_n}{n}t)}{\pi_0}\right] \\
&= n\log\mathbb{E}\frac{1}{\pi_0}\int e^{a_n n^{-1}tx}X_0(dx) \\
&= n\log\mathbb{E}\frac{1}{m_0}\int\left(\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{a_n}{n}t\right)^k x^k\right)X_0(dx) \\
&= n\log\left(1+\sum_{k=2}^{\infty}\frac{1}{k!}\left(\frac{a_n}{n}t\right)^k\mathbb{E}\frac{1}{\pi_0}\int x^kX_0(dx)\right) \\
&= n\log\left(1+\frac{1}{2}\left(\frac{a_n}{n}\right)^2t^2\sigma^2+o(\left(\frac{a_n}{n}\right)^2)\right).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\frac{n}{a_n^2}\lambda_n(\frac{a_n^2}{n}t)=\lim_{n\to\infty}\frac{n^2}{a_n^2}\log\left(1+\frac{1}{2}\left(\frac{a_n}{n}\right)^2t^2\tilde{\sigma}^2+o(\left(\frac{a_n}{n}\right)^2)\right)=\frac{1}{2}\sigma^2t^2.$$

□

If we consider the probability measures $\frac{\mathbb{E}Z_n(a_n\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$, then by a similar argument to the proof of Theorem 5.2, we obtain the following theorem.

Theorem 5.3 (Moderate deviation principle for annealed means $\frac{\mathbb{E}Z_n(a_n\cdot)}{\mathbb{E}Z_n(\mathbb{R})}$)**.** *Assume that ξ_n are i.i.d.. Write $\pi_0 = m_0(0)$. If $\mathbb{E}\sum_{i=1}^{N}e^{\delta|L_i|}<\infty$ for some $\delta<0$ and $\mathbb{E}\sum_{i=1}^{N}L_i=0$, then the sequence of finite measures $A\mapsto\frac{\mathbb{E}Z_n(a_nA)}{\mathbb{E}Z_n(\mathbb{R})}$ satisfies a principle of moderate deviation: for each measurable subset A of $\mathbb{R}$,*

$$\begin{aligned}
-\frac{1}{2\tilde{\sigma}^2}\inf_{x\in A^o}x^2 &\le \liminf_{n\to\infty}\frac{n}{a_n^2}\log\frac{\mathbb{E}Z_n(a_nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})} \\
&\le \limsup_{n\to\infty}\frac{n}{a_n^2}\log\frac{\mathbb{E}Z_n(a_nA)}{\mathbb{E}_\xi Z_n(\mathbb{R})}\le-\frac{1}{2\tilde{\sigma}^2}\inf_{x\in\bar{A}}x^2, \qquad (5.10)
\end{aligned}$$

where $\tilde{\sigma}^2=\frac{1}{\mathbb{E}m_0}\mathbb{E}\sum_{i=1}^{N}L_i^2$, and A^o denotes the interior of A and $\bar{A}$ its closure.

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